New Solitary wave Solutions of Nonlinear Partial Differential Equations

Anwar Ja'afar Mohamad Jawad

Abstract - In this paper, we used the proposed Tan-Cot function method for establishing a traveling wave solution to nonlinear partial differential equations. The method is used to obtain new solitary wave solutions for various types of nonlinear partial differential equations such as, Cassama-Holm equation, Broer-Kaup system, and KdV Evolutionary System, which are the important Soliton equations. Proposed method has been successfully implemented to establish new solitary wave solutions for the nonlinear PDEs.

Index Terms- Nonlinear PDEs, Exact Solutions, tan-Cot function method, Cassama-Holm equation, Broer-Kaup system, KdV Evolutionary System.

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1. INTRODUCTION

Many models in mathematics and physics are described by nonlinear differential equations. Nowadays, research in physics devotes much attention to nonlinear partial differential evolution model equations, appearing in various fields of science, especially fluid mechanics, solid-state physics, plasma physics, and nonlinear optics. Large varieties of physical, chemical, and biological phenomena are governed by nonlinear partial differential equations. One of the most exciting advances of nonlinear science and theoretical physics has been the development of methods to look for exact solutions of nonlinear partial differential equations. Exact solutions to nonlinear partial differential equations play an important role in nonlinear science, especially in nonlinear physical science since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed.

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A variety of powerful methods, such as, tanhsech method {Malfliet [1], Khater et al. [2], Wazwaz [3]}, extended tanh method {El-Wakil et al.[4], Fan[5], Wazwaz [6]}, hyperbolic function method, Xia and Zhang [7], Yusufoglu and Bekir [8], Jacobi elliptic function expansion method Inc and Ergut [9], F-expansion method Zhang [10], and the First Integral method Fen[11], Ding and Li [12] .The sine-cosine method {Mitchell [13], Parkes [14], Khater et all [2]} has been used to solve different types of nonlinear systems of PDEs.

In this paper, we applied the tan-cot function method to solve Cassama-Holm equation, Broer-Kaup system, and KdV Evolutionary System, given respectively by:

$$u_t + 2a\, u_x - u_{xxt} + 3\, u\, u_x - 2u_x u_{xx} - u\, u_{xxx} = 0 \qquad (1)$$

and the the following systems

$$u_t + u\, u_x + v_x = 0 \qquad\qquad (2)$$

$$v_t + u_x + (u\, v)_x + u_{xxx} = 0 \qquad\qquad (3)$$

and

$$u_t + 3v_{xx} = 0 \qquad\qquad (4)$$

$$v_t - u_{xx} - 4\, u^2 = 0 \qquad\qquad (5)$$

2. THE TAN-COT FUNCTION METHOD

Consider the nonlinear partial differential equation in the form

$$F\big(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{xy}, u_{yy}, \ldots\ldots\ldots\ldots\big) = 0 \qquad (6)$$

where u(x, y, t) is a traveling wave solution of

nonlinear partial differential equation Eq. (6). We use the transformation,

$$u(x, y, t) = f(\xi) \tag{7}$$

Where

$$\xi = kx + y - \lambda t \tag{8}$$

where k, and λ are real constants. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -\lambda \frac{d}{d\xi}(.) \ , \ \frac{\partial}{\partial x}(.) = k \frac{d}{d\xi}(.) \ ,$$

$$\frac{\partial}{\partial y}(.) = \frac{d}{d\xi}(.) \tag{9}$$

Using Eq. (8) to transfer the nonlinear partial differential equation Eq. (6) to nonlinear ordinary differential equation

$$Q(f, f', f'', f''', \ldots \ldots \ldots \ldots .) = 0 \tag{10}$$

The ordinary differential equation (10) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the form [15]:

$$f(\xi) = \alpha \tan^\beta(\mu\xi) \ , \qquad |\xi| \le \frac{\pi}{2\mu}$$

or in the form

$$f(\xi) = \alpha \cot^\beta(\mu\xi) \ , \qquad |\xi| \le \frac{\pi}{2\mu} \tag{11}$$

Where α , μ, and β are parameters to be determined, μ is the wave number. We use

$$f(\xi) = \alpha \tan^\beta(\mu\xi)$$

$$f'(\xi) = \alpha \beta \mu \ [\tan^{\beta - 1}(\mu\xi) + \tan^{\beta + 1}(\mu\xi)] \tag{12}$$

$$f'(\xi) = \alpha \beta\mu^2[(\beta - 1) \tan^{\beta - 2}(\mu\xi) + 2\beta \tan^\beta(\mu\xi) + (\beta + 1) \tan^{\beta + 2}(\mu\xi)]$$

$$f''(\xi) = \beta\mu^3\alpha[(\beta - 1)(\beta - 2) \tan^{\beta - 3}(\mu\xi) + (3\beta^2 - 3\beta + 2) \tan\beta{-}1 \ \mu\xi{+}\beta{+}1\beta{+}2\tan\beta\mu\xi \ + 2\beta2 \ \tan\beta{+}1 \mu\xi{+}(\beta{+}1)(\beta{+}2) \tan\beta{+}2\mu\xi]$$

and their derivative. Or use

$$f(\xi) = \alpha \cot^\beta(\mu\xi)$$

$$f'(\xi) = -\alpha \beta \mu \ [\cot^{\beta - 1}(\mu\xi) + \cot^{\beta + 1}(\mu\xi)] \tag{13}$$

$$f'(\xi) = \alpha \beta\mu^2[(\beta - 1) \cot^{\beta - 2}(\mu\xi) + 2\beta \cot^\beta(\mu\xi) + (\beta + 1) \cot^{\beta + 2}(\mu\xi)]$$

and so on. We substitute (12) or (13) into the reduced equation (10), balance the terms of the tan functions when (12) are used, or balance the terms of the cot functions when (13) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\tan^k(\mu\xi)$ or $\cot^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown's α , μ and β, and solve the subsequent system.

3. APPLICATIONS

3.1 Cassama-Holm equation

Consider the Cassama-Holm equation

$$u_t + 2a \, u_x - u_{xxt} + 3 \, u \, u_x - 2u_x u_{xx} - u \, u_{xxx} = 0 \tag{14}$$

This equation was studied by Marwan [16]. He used sine-cosine method to establish an exact solution.

We introduce the transformation $\xi = k(x - \lambda t)$ where k, and λ are real constants. Equation (14) transforms to the ODE:

$$(2a - \lambda)u' + \lambda \, k^2 \, u''' + \frac{3}{2} \, (u^2)' - k^2 \, [\, (u'^{\,2})' + u \, u'''] = 0 \tag{15}$$

Equation (15) can be written as

$$(2a - \lambda)u' + \lambda \, k^2 \, u''' + \frac{3}{2} \, (u^2)' - k^2 \, \left[\frac{1}{2}(u'^{\,2})' + u \, u'''\right] = 0 \tag{16}$$

Integrating (16) with zero constant, we get

$$(2a - \lambda)u + \lambda \, k^2 \, u'' + \frac{3}{2} \, u^2 - k^2 \, \left[\frac{1}{2}u'^{\,2} + u \, u''\right] = 0 \tag{17}$$

Seeking the solution in (12)

$$(2a - \lambda)\alpha \tan^\beta(\mu\xi) + \lambda \, k^2\mu^2 \, \alpha \, \{(\beta - 1) \, \tan^{\beta - 2}(\mu\xi) + 2\beta \, \tan^\beta \, (\mu\xi) + (\beta + 1) \, \tan\beta{+}2\mu\xi - k2\alpha \, \beta\mu2\alpha \, (\beta{-}1) \, \tan2\beta{-}2\mu\xi{+} \, 2\beta \, \tan2\beta \, \mu\xi{+}(\beta{+}1) \tan2\beta{+}2\mu\xi \, {+}32 \, \alpha2 \tan2\beta\mu\xi - \, 12 \, k2\alpha2\beta2 \, \mu2 \, [\, \tan^{2\beta - 2}(\mu\xi) + 2 \, \tan^{2\beta} + \, \tan^{2\beta + 2}(\mu\xi)] \, = 0 \tag{18}$$

Equating the exponents and the coefficients of each pair of the tan functions we find the following algebraic system:

$$2\beta = \beta + 2 \ \rightarrow \beta = 2 \tag{19}$$

Substituting Eq. (19) into Eq. (18) to get the following system of equations:

$$- k^2\alpha \, \beta\mu^2\alpha(\beta - 1) - \frac{1}{2} k^2\alpha^2\beta^2 \, \mu^2 = 0$$

$$\lambda \, k^2\beta(\beta + 1)\mu^2 \, \alpha - 2\beta \, k^2\alpha \, \beta\mu^2\alpha + \frac{3}{2} \, \alpha^2 - k^2\alpha^2\beta^2 \, \mu^2 = 0 \tag{20}$$

$$(2a - \lambda)\alpha + 2\beta \, \lambda \, k^2\beta\mu^2 \, \alpha = 0$$

Solving (20) then:

$$\alpha = 24 \, k^2\mu^2a/[1 - 8 \, k^2 \, \mu^2][20 \, k^2\mu^2 - 3\,]$$

$$\lambda = 2a/[1 - 8 \, k^2 \, \mu^2] \tag{21}$$

Then by substituting Eq.(21) into Eq.(12), the exact soliton solution of equation (14) can be written in the form

$$u(x, t) = 24 \, k^2\mu^2a/[1 - 8 \, k^2 \, \mu^2][20 \, k^2\mu^2 - 3 \, \tan2 \, \mu \, kx - 2at/[1{-}8 \, k2 \, \mu2] \tag{22}$$

For $\mu = k = a = 1$ Eq.(22) becomes

$$u(x, t) = -(24/119) \tan^2 (x + 2t/7) \tag{23}$$

$u(x, t)$ in (23) is represented in Figure (1) for $0 \le x \le 10$ and $0 \le t \le 1$.

3.2 Broer-kaup System

Consider the Broer-Kaup system;

$$u_t + u \, u_x + v_x = 0 \tag{24}$$

$$v_t + u_x + (u \, v)_x + u_{xxx} = 0 \tag{25}$$

This equation was studied by Marwan et al [16]. They used sine-cosine method to establish an exact solution.
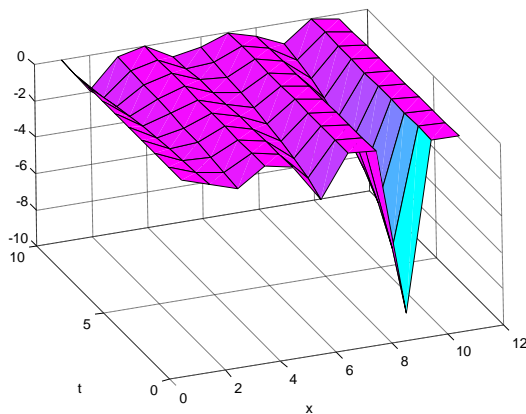

Fig. (1) represents $u(x,t)$ in (23) for $0 \le x \le 10$ and $0 \le t \le 1$.

Using the wave variable $\xi = k(x - \lambda t)$ carries (24), and (25) into the ODEs

$$-\lambda\, u' + \frac{1}{2}(u^2)' + v' = 0 \qquad (26)$$

$$-\lambda\, v' + u' + (uv)' + k^2 u''' = 0 \qquad (27)$$

integrating the ODEs and setting the constant of integration to zero. From (26) we have

$$-\lambda\, u + \frac{1}{2}u^2 + v = 0 \qquad (28)$$

Then

$$v = \lambda\, u - \frac{1}{2}u^2 \qquad (29)$$

From (29) we have

$$-\lambda v + u + uv + k^2 u'' = 0 \qquad (30)$$

Substitute (29) into (30) gives:

$$(1 - \lambda^2)u + \frac{3}{2}\lambda\, u^2 - \frac{1}{2}u^3 + k^2 u'' = 0 \qquad (31)$$

Seeking the method in (12)

$$(1-\lambda^2)\alpha \tan^\beta(\mu\xi) + \frac{3}{2}\lambda\, \alpha^2 \tan^{2\beta}(\mu\xi) - \frac{1}{2}\alpha^3 \tan^{3\beta}(\mu\xi) +$$
$$k^2\alpha\beta\mu^2 \,[(\beta-1)\,\tan^{\beta-2}(\mu\xi) + 2\beta\,\tan^\beta(\mu\xi) + (\beta+1)\tan^{\beta+2}(\mu\xi)] \;= 0$$

$$(32)$$

Equating the exponents and the coefficients of each pair of the cot function, we obtain

$$3\beta = \beta + 2 \text{ , so that } \beta = 1 \qquad (33)$$

Substitute (33) into (32) give the following system of equations

$$(1-\lambda^2)\alpha + 2\, k^2\alpha\,\mu^2 = 0$$
$$4k^2\alpha\mu^2 - \alpha^3 = 0 \qquad (34)$$

Solving (34) will gives:

$$\mu = \mp\sqrt{\frac{\lambda^2-1}{2\,k^2}} \; , \; \alpha = \mp\sqrt{2}\,\sqrt{\lambda^2-1} \qquad (35)$$

Then by substituting Eq.(35) into Eq.(12), the exact soliton solution of the system of equations (24) and (25) can be written in the form

$$u(x,t) = \mp\sqrt{2}\,\sqrt{\lambda^2-1}\,\tan\left(\mp\sqrt{\frac{\lambda^2-1}{2}}\,(x-\lambda t)\right) \qquad (36)$$

$$v(x,y) = \mp\sqrt{2}\,\sqrt{\lambda^2-1}\,\tan\left(\mp\sqrt{\frac{\lambda^2-1}{2}}\,(x-\lambda t)\right)\Big[\lambda - \frac{\sqrt{\lambda^2-1}}{\sqrt{2}}\,\tan\left(\mp\sqrt{\frac{\lambda^2-1}{2}}\,(x-\lambda t)\right)\Big]$$

$$(37)$$

For $\lambda = 3/2$ then

$$u(x,t) = \mp\sqrt{\frac{5}{2}}\,\tan\left(\mp\frac{1}{2}\sqrt{\frac{5}{2}}\,(x-\frac{3}{2}t)\right)$$

$$(38)$$

$$v(x,y) = \mp\sqrt{\frac{5}{2}}\,\tan\left(\mp\frac{1}{2}\sqrt{\frac{5}{2}}\,(x-\frac{3}{2}t)\right)\Big[\lambda - \frac{1}{2}\sqrt{\frac{5}{2}}\,\tan\left(\mp\frac{1}{2}\sqrt{\frac{5}{2}}(x- \frac{3}{2}t)\right)$$

$$(39)$$

$u(x,t)$, and $v(x,t)$ in (38) and (39) respectively are shown in figures (2) and (3) respectively for $0 \le x \le 10$ and $0 \le t \le 1$.
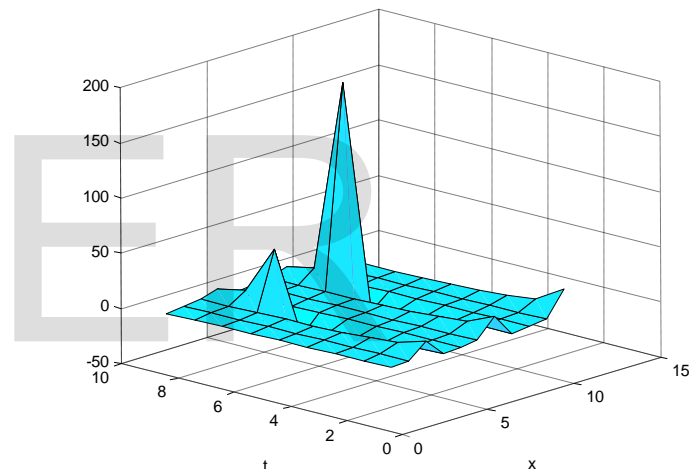

Fig. (2) represents $u(x,t)$ for $0 \le x \le 10$ and $0 \le t \le 1$.

3.3 KdV Evolutionary System

Consider the two-component KdV evolutionary System of Order 2:

$$u_t + 3v_{xx} = 0 \qquad (40)$$
$$v_t - u_{xx} - 4\, u^2 = 0 \qquad (41)$$

This equation was studied by Marwan et al [16]. They used sine-cosine method to establish an exact solution.
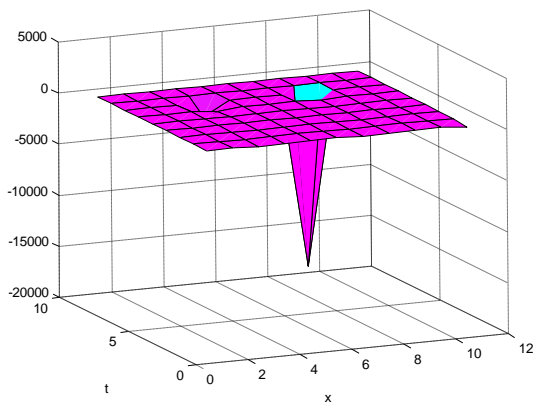
Fig. (3) represents $v(x, t)$ for $0 \leq x \leq 10$ and $0 \leq t \leq 1$.

Using the wave variable $\xi = k(x - \lambda t)$ carries (40), and (41) into the ODEs

$$-\lambda u' + 3 k v'' = 0 \qquad (42)$$

$$-\lambda k v' - k^2 u'' - 4 u^2 = 0 \qquad (43)$$

integrating the ODE (42) and setting the constant of integration to zero. we have

$$-\lambda u + 3 k v' = 0 \qquad (44)$$

Let:

$$u = \alpha_1 \tan^{\beta_1}(\mu\xi) \ , \ v = \alpha_2 \tan^{\beta_2}(\mu\xi) \qquad (45)$$

Substitute (45) and their derivatives then (42) becomes:

$$-\lambda \alpha_1 \tan^{\beta_1}(\mu\xi) + 3 k \alpha_2 \beta_2 \mu \ [\tan^{\beta_2 - 1}(\mu\xi) + \tan^{\beta_2 + 1}(\mu\xi)] = 0 \qquad (46)$$

$$-\lambda k \alpha_2 \beta_2 \mu \ [\tan^{\beta_2 - 1}(\mu\xi) + \tan^{\beta_2 + 1}(\mu\xi)] - k^2\alpha_1 \beta_1\mu^2[(\beta_1 - 1) \tan^{\beta_1 - 2}(\mu\xi) + 2\beta_1 \tan^{\beta_1} (\mu\xi) + (\beta_1 + 1) \tan^{\beta_1 + 2}(\mu\xi)] - 4 \alpha_1{}^2 \tan^{2\beta_1}(\mu\xi) = 0$$

$$\qquad (47)$$

Equating the exponents and the coefficients of each pair of the cot function, we obtain

$\beta_2 + 1 = \beta_1$, so that $\beta_2 - \beta_1 = -1$

$2\beta_1 = \beta_1 + 2$, so that $\beta_1 = 2, \beta_2 = 1$

Substitute $\beta_1 = 2$, and $\beta_2 = 1$ into (46), and (47) give the following system of equations

$-\lambda \alpha_1 + 3 k \mu \alpha_2 = 0$

$\lambda \alpha_2 + 2\alpha_1 k\mu = 0$

$$3k^2\mu^2 + 2 \alpha_1 = 0 \qquad (48)$$

Solving (48) will gives:

$\lambda = \mp i \sqrt{6} k \mu , \alpha_1 = - \frac{3 k^2 \mu^2}{2} ,$

$$\alpha_2 = \pm i \sqrt{\frac{3}{2}} k^2 \mu^2 \qquad (49)$$

Then by substituting Eq.(49) into Eq.(45), the exact soliton solution of the system of equations (40) and (41) can be written in the form

$$u(x, t) = - \frac{3 k^2 \mu^2}{2} \cot^2(\mu k(x \pm i \sqrt{6} k \mu t))$$

$$\qquad (50)$$

$$v(x, t) = \pm i \sqrt{\frac{3}{2}} k^2 \mu^2 \cot(\mu k(x \pm i \sqrt{6} k \mu t)) \qquad (51)$$

For $k = \mu = 1$, then $\lambda = \mp i \sqrt{6}$

$$u(x, t) = - \frac{3}{2} \cot^2(x \pm i \sqrt{6} t) \qquad (52)$$

$$v(x, t) = \pm i \sqrt{\frac{3}{2}} \cot(x \pm i \sqrt{6} t) \qquad (53)$$

4. CONCLUSION

In this paper, the tan-cot function method has been successfully method has been successfully implemented to establish new solitary wave solutions for various types of nonlinear PDEs. We can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

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